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Stäckel systems generating coupled KdV hierarchies and their finite-gap and rational solutions

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Abstract

We show how to generate coupled KdV hierarchies from Stäckel separable systems of Benenti type. We further show that the solutions of these Stäckel systems generate a large class of finite-gap and rational solutions for cKdV hierarchies. Most of these solutions are new.

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1. Introduction

In [1], we presented a systematic method of passing from Stäckel separable systems to infinite hierarchies of commuting nonlinear evolutionary PDEs. We presented the idea in a concrete case of Stäckel systems of Benenti type. In this paper we recognize the obtained hierarchies as the well-known coupled Korteweg–de Vries (cKdV) hierarchies of Antonowicz and Fordy [2] written in a different representation. We also clarify and significantly simplify the approach developed in [1]. The main idea of the paper is however to present a new method of generating solutions of soliton hierarchies from solutions of the related Stäckel systems.

From the very beginning of development of the theory of integrable systems in the late 1960s major efforts have been put into constructing various ways of finding their solutions. Among many others, a possible way of finding solutions of integrable systems is through various kinds of symmetry reduction, where one starts from an infinite-dimensional integrable system and obtains after such reduction an integrable ODE. If one then succeeds in solving this ODE (for example by finding separation coordinates in the case of Hamiltonian systems) one can then reconstruct the corresponding particular solutions of the integrable PDE. This method originated in [3] and has been developed in [2, 4, 5] and later in a large number of papers. In this paper, we present an opposite approach in the sense that we start with large classes of Stäckel systems written in separation coordinates so that their solutions are explicitly known and in few steps we construct from these Stäckel systems an infinite hierarchy of commuting evolutionary PDEs while the solutions of the considered Stäckel systems become particular multi-time solutions of the systems of the obtained hierarchy. In this way we produce both well-known and new finite-gap-type solutions of the KdV hierarchy as well as new rational and finite-gap solutions of cKdV hierarchies. In many cases this method also leads to implicit solutions of cKdV hierarchies.

Rational solutions of KdV were first obtained in [6]. In [7] it has been demonstrated that the rational solutions of KdV originate as long-wave limit (as the wave number $k \to 0$) of multi-soliton solutions, obtained in this case directly through bilinear method of Hirota. In [8] rational solutions of KdV and other soliton equations have been obtained through the Painleve property. One can also produce rational solutions of KdV by using Yablonskii–Vorob'ev polynomials [9]. Our method is novel in that it is based on a different principle, it produces multi-time solutions (solutions that contain an arbitrary number of times of the hierarchy) and moreover it produces rational solutions of coupled (multi-component) hierarchies, which is to our knowledge not present in the literature.

This paper is organized as follows. In section 2 we briefly describe the starting point of our considerations, that is Stäckel separable systems of Benenti type, including their general solution. In section 3 we relate with our Stäckel systems a class of weakly-nonlinear semi-Hamiltonian systems (i.e. a class of hydrodynamic-type systems) that are reductions of the so-called universal hierarchy [10]. These systems are in our formulation defined by Killing tensors of Stäckel metrics. In section 4 we explicitly construct hierarchies of commuting evolutionary PDEs and present a transformation that maps these hierarchies onto the well-known cKdV hierarchies of Antonowicz and Fordy [2]. This section contains the main result of this paper, i.e. theorem 8, that produces large families of solutions to our coupled hierarchies. Finally, section 5 is devoted to studying specific classes of solutions that we call zero-energy solutions that contain both rational and implicit solutions of our hierarchies. We also discuss how our solutions are related to what can be found in the literature.

2. Stäckel systems of Benenti type

Let us consider a set of canonical (Darboux) coordinates $(\lambda, \mu) = (\lambda_1 \ldots, \lambda_n, \mu_1, \ldots, \mu_1)$ on a $2n$-dimensional Poisson manifold M. Relations of the form

$$\varphi_i(\lambda_i, \mu_i, a_1, \ldots, a_n) = 0, \qquad i = 1, \ldots, n, \qquad a_i \in \mathbf{R} \tag{1}$$

(each involving only one pair λ_i, μ_i of canonical coordinates) are called separation relations [11] provided that $\det\left(\frac{\partial \varphi_i}{\partial a_j}\right) \neq 0$. Resolving (locally) equations (1) with respect to a_i we obtain

$$a_i = H_i(\lambda, \mu), \qquad i = 1, \ldots, n \tag{2}$$

with some new functions (Hamiltonians) $H_i(\lambda, \mu)$ that in turn define n canonical Hamiltonian systems on M:

$$\lambda_{t_i} = \frac{\partial H_i}{\partial \mu}, \qquad \mu_{t_i} = -\frac{\partial H_i}{\partial \lambda}, \qquad i = 1, \ldots, n \tag{3}$$

(here and in what follows the subscript denotes derivative with respect to the subscript variable). From this setting it follows immediately that the Hamiltonians H_i Poisson commute. The corresponding Hamilton–Jacobi equations for all Hamiltonians H_i are separable in the (λ, μ)-variables since they are algebraically equivalent to the separation relations (1).

In this paper we consider a special but important class of separation relation:

$$\sum_{j=1}^{n} a_j \lambda_i^{n-j} = A f(\lambda_i) \mu_i^2 + B \gamma(\lambda_i), \qquad i = 1, \ldots, n, \tag{4}$$

where A and B are two real constants to be specified later. Note that since the functions γ and f do not depend on i the relations (4) can in fact be considered as n copies of a curve—the so-called *separation curve* in the λ–μ plane. The Hamiltonian systems obtained from this class of separation relations have been widely studied and are also known as Benenti systems. Benenti systems constitute the simplest, but still very wide, class of all possible Stäckel separable systems. It can be shown [12] that this class contains all quadratic in momenta Stäckel separable systems since all other systems of this type are constructed from (4) by appropriate generalized Stäckel transforms and related reciprocal transformations.

Let us now recall some established facts about Benenti systems. The relations (4) are linear in the coefficients a_i. Solving these relations with respect to a_i we obtain

$$a_i = A\mu^T K_i G(f)\mu + BV_i(\gamma) \equiv H_i, \qquad i = 1, \ldots, n, \tag{5}$$

where we use the notation $\lambda = (\lambda_1, \ldots, \lambda_n)^T$ and $\mu = (\mu_1, \ldots, \mu_n)^T$. Functions H_i defined as the right-hand-sides of the solution (5) can be (locally) interpreted as n quadratic in momenta μ Hamiltonians on the phase space $M = T^*\mathcal{Q}$ cotangent to a Riemannian manifold $\mathcal{Q}$ equipped with the contravariant metric tensor $G(f)$ depending on the function f only. As mentioned above, these Hamiltonians are in involution with respect to the canonical Poisson bracket on $T^*\mathcal{Q}$. Moreover, they are separable in the sense of Hamilton–Jacobi theory since they by the very definition satisfy Stäckel relations (1). The objects K_i in (5) can be interpreted as $(1, 1)$-type Killing tensors on $\mathcal{Q}$ related to the family of metrics $G(f)$. The scalar functions $V_i(\gamma)$ depend only on the function γ and can be considered as separable potentials. Further, the metric tensor G and all the Killing tensors K_i are diagonal in λ-variables. More specifically, in λ-variables they attain the form

$$G(f) = \operatorname{diag}\left(\frac{f(\lambda_1)}{\Delta_1}, \ldots, \frac{f(\lambda_n)}{\Delta_n}\right) \qquad \text{with} \quad \Delta_i = \prod_{j\neq i}(\lambda_i - \lambda_j) \tag{6}$$

and

$$K_i = -\operatorname{diag}\left(\frac{\partial q_i}{\partial\lambda_1}, \cdots, \frac{\partial q_i}{\partial\lambda_n}\right) \qquad i = 1, \ldots, n$$

respectively. Here and below $q_i = q_i(\lambda)$ are Viète polynomials (signed symmetric polynomials) in λ:

$$q_i(\lambda) = (-1)^i \sum_{1\leqslant s_1<s_2<\ldots<s_i\leqslant n} \lambda_{s_1}\ldots\lambda_{s_i}, \qquad i = 1, \ldots, n \tag{7}$$

that can also be considered as new coordinates on the Riemannian manifold $\mathcal{Q}$ (we will then refer to them as Viète coordinates). Note that the Killing tensors do not depend on a particular choice of f and γ.

Remark 1. The general n-time (simultaneous) solution for the Hamilton equations (3) associated with all the Hamiltonians (5) attains the form

$$t_i + c_i = \pm\frac{1}{2\sqrt{A}}\sum_{r=1}^{n}\int\frac{\lambda_r^{n-i}}{\sqrt{f(\lambda_r)\left(\sum_{j=1}^{n}a_j\lambda_r^{n-j} - B\gamma(\lambda_r)\right)}}\,\mathrm{d}\lambda_r, \qquad i = 1, \ldots, n. \tag{8}$$

To see this it is enough to integrate the related Hamilton–Jacobi problem. Now, with every Hamiltonian H_i in (5) we can associate the corresponding inverse Legendre mapping (fiber derivartive) $\mathcal{L}_i^{-1} : T^*\mathcal{Q} \to T\mathcal{Q}$ given in the natural coordinates (λ, μ) on $T^*\mathcal{Q}$ and (λ, λ_{t_i}) on $T\mathcal{Q}$ respectively by

$$\mathcal{L}_i^{-1}(\lambda, \mu) = (\lambda, 2AK_iG\mu) = (\lambda, \lambda_{t_i}).$$

Performing n corresponding Legendre transforms we obtain n Lagrangians $L_i : TQ \to \mathbf{R}$ given explicitly by

$$L_i(\lambda, \lambda_{t_i}) = \frac{1}{4A}\lambda_{t_i}^T g K_i^{-1}\lambda_{t_i} - BV_i(\gamma), \tag{9}$$

where $g = G^{-1}$ is the corresponding covariant metric tensor. These Lagrangians give rise to n systems of Euler–Lagrange equations

$$E_{t_i}(L_i) = 0, \qquad i = 1, \ldots, n, \tag{10}$$

where E_{t_i} is the Euler–Lagrange operator with respect to the independent variable t_i. By construction, the solutions (8) are also general solutions for all the Euler–Lagrange equations (10). This means that for a particular i the general solution of the Euler–Lagrange equation $E_{t_i}(L_i) = 0$ is given by (8) where t_j is constant for $j \neq i$.

3. Dispersionless Killing systems of Benenti type

We have shown above that the separation relations (4) lead to separable Liouville systems (Benenti systems) and we also presented its general solutions. It turns out that with the set of Killing tensors K_i of any Benenti system we can also relate a set of first-order evolutionary PDEs.

Definition 2 [13]. *For any fixed $j \in \{1, \ldots, n\}$, any of the following n systems of evolutionary PDEs of the form*

$$\lambda_{t_i} = K_i K_j^{-1}\lambda_{t_j} \equiv Z_{ij}(\lambda, \lambda_{t_j}), \qquad i = 1, \ldots, n \tag{11}$$

(where $\lambda = (\lambda_1, \ldots, \lambda_n)^T$) with K_i being Killing tensors of a Benenti system (5) will be called a dispersionless Killing system of Benenti type.

The chosen variable t_j in (11) plays the role of a space variable while the remaining variables t_i should then be considered as evolution parameters (times). Equations (11) constitute a set of n integrable dispersionless equations, belonging to the class of so-called *weakly nonlinear* hydrodynamic-type systems, i.e. that are semi-Hamiltonian in the sense of Tsarev [14, 15] and linearly degenerate [16], where the variables λ_i are the Riemann invariants for the system. For a chosen j, the systems (11) can be considered as n vector fields Z_{ij} (with a fixed j) on some infinite-dimensional function space $\mathcal{M}_j$ of functions $(\lambda_1(t_j), \ldots, \lambda_n(t_j))$. It can be shown that

Proposition 3. *On $\mathcal{M}_j$ (i.e. for a fixed j) the vector fields Z_{ij} commute:*

$$[Z_{ij}, Z_{kj}] = 0 \qquad i, k = 1, \ldots, n.$$

The following proposition is crucial for our study:

Proposition 4. *Every mutual solution $\lambda(t_1, \ldots, t_n)$ (given by (8)) of all Hamiltonian systems (3) with Hamiltonians of Benenti type (5) is also a solution of all corresponding Killing system in (11).*

Proof. For the class of Benenti systems (5) the spatial part of (3) attains the form

$$\lambda_{t_i} = \frac{\partial H_i}{\partial \mu} = 2AK_i G\mu, \qquad i = 1, \ldots, n. \tag{12}$$

So, for any fixed $j \in \{1, \ldots, n\}$ we can eliminate the momenta μ from (12). This yields (11). □

Thus, all the solutions (8) are also solutions of all n Killing systems in (11). Moreover, we have

Theorem 5. *The n-time general solution of all the Killing systems in (11) is given by*

$$t_i + c_i = \sum_{r=1}^{n} \int \frac{\lambda_r^{n-i}}{\varphi_r(\lambda_r)} \, \mathrm{d}\lambda_r, \qquad i = 1, \ldots, n \tag{13}$$

(where φ_r are arbitrary functions of one variable).

The proof of this statement can be found in [16]. Thus (13) indeed encompasses all the solutions (8). We also see that any solution (13) can be written in the form (8) (with appropriately chosen f, γ, a_j, A and B) so that on a given surface of fixed values of all a_i (for example for zero-energy solutions, see below) with any such solution we can associate infinitely many corresponding Stäckel systems (3) sharing the same solution.

Consider now solutions (8) as a specific class of solutions of (11). Since this class of solutions—by construction—satisfies all the Euler–Lagrange equations (10) we can treat these equations as additional bonds that these solutions satisfy. We can therefore use these bonds to express some variables λ_i by other λs. Thus, within the class (8) of solutions of the Killing system (11) we can perform a variable elimination (reparametrization) that turns (11) into entirely new sets of evolutionary PDEs. Below we demonstrate that in some carefully chosen cases this reparametrization turns systems (11) into systems with dispersion (soliton hierarchies) with the solution (8) being also a solution of these new systems with dispersion. Specifically, we will produce this way all coupled KdV hierarchies as well as new interesting classes of their solutions: finite-gap and rational solutions. We will also demonstrate that our hierarchies indeed are the well-known cKdV hierarchies obtained by Antonowicz and Fordy through the energy-dependent Schrödinger spectral problem [2].

From now on we will assume that

$$f = \lambda^m, \qquad \gamma = \lambda^k, \qquad m, k \in \mathbf{Z} \tag{14}$$

so that (4) attains the form

$$\sum_{j=1}^{n} a_j \lambda_i^{n-j} = A\lambda_i^m \mu_i^2 + B\lambda_i^k, \qquad i = 1, \ldots, n. \tag{15}$$

We will denote the metric associated with $f = \lambda^m$ through (6) by $G^{(m)}$. It can be shown that for $m = 0, \ldots, n$ the metric $G^{(m)}$ is of zero curvature while the metric $G^{(n+1)}$ has a non-zero constant curvature. The separable potentials associated with $\gamma = \lambda^k$ will be denoted by $V_i^{(k)}$. The family of separable potentials $V_i^{(k)}$ can be constructed recursively [17] by

$$V_i^{(k+1)} = V_{i+1}^{(k)} - q_i V_1^{(k)} \qquad \text{with} \qquad V_i^{(0)} = \delta_{in}, \tag{16}$$

where we put $V_i^{(k)} = 0$ for $i < 0$ or $i > n$. The first potentials are trivial: $V_i^{(k)} = \delta_{i,n-k}$ for $k = 0, 1, \ldots, n-1$. The first nontrivial potential is $V_i^{(n)} = -q_i$, For $k > n$ the potentials $V_i^{(k)}$ become complicated polynomial functions of λ.

The Lagrangians (9) for any specific choice of m and n are denoted as $L_i^{n,m,k}$ so that $L_i^{n,m,k} = \frac{1}{4A} \lambda_{t_i}^T g^{(m)} K_i^{-1} \lambda_{t_i} - BV_i^{(k)}$.

From now on we will choose the representation $j = 1$ in (11) so that the variable t_1 plays the role of the space variable. We denote therefore this variable as x: $t_1 = x$. The case of higher j is not discussed in this paper. Thus, we consider the Killing systems of the form

$$\lambda_{t_i} = K_i \lambda_x \equiv Z_i^n(\lambda, \lambda_x), \qquad i = 1, \ldots, n, \tag{17}$$

where the new upper index n in the ith vector field Z_i denotes the number of its components; note also that the second lower index in Z is now always 1 and can therefore be omitted. Also, from now on we denote the Lagrangian $L_1^{n,m,k}$ simply as $L^{n,m,k}$ (to avoid the unnecessary index) so that

$$L^{n,m,k} = \frac{1}{4A}\lambda_x^T g^{(m)}\lambda_x - BV_1^{(k)}, \qquad i = 1, \dots, n. \tag{18}$$

In order to perform the aforementioned elimination procedure we will first pass to Viète coordinates (7). The Killing systems (17) are tensorial so in Viète coordinates they have the form $q_{t_r} = K_r(q)q_x$ or, explicitly

$$\frac{\mathrm{d}}{\mathrm{d}t_i}q_j = (q_{j+i-1})_x + \sum_{k=1}^{j-1}(q_k(q_{j+i-k-1})_x - q_{j+i-k-1}(q_k)_x) \equiv \left(Z_i^n[q]\right)^j, \qquad i, j = 1, \dots, n, \tag{19}$$

where $q_\alpha = 0$ as soon as $\alpha > n$ and $\left(Z_i^n[q]\right)^j$ denotes the jth component of the vector field $Z_i^n[q] \equiv Z_i^n(q, q_x)$ (here and below the symbol $f[q]$ will denote a differential function of q, that is a function depending on q and a finite number of its derivatives). One can see that $\left(Z_i^n[q]\right)^j = \left(Z_j^n[q]\right)^i$ for all $i, j = 1, \dots, n$. Further, in Viète coordinates the Lagrangian (18) takes the form

$$L^{n,m,k} = L^{n,m,k}(q, q_x) = \frac{1}{4A}q_x^T g^{(m)} q_x - BV_1^{(k)}, \qquad i = 1, \dots, n, \tag{20}$$

where $g_{ij}^{(m)} = V_1^{(2n-m-i-j)}$ [1]. The Euler–Lagrange operator E_{t_1} in (10) will be simply denoted as E, so in Viète coordinates

$$E = (E_1, \dots E_n), \qquad E_i = \frac{\delta}{\delta q_i}.$$

Theorem 6. *Lagrangian (20) satisfies the following symmetry relations:*

(1) for $\alpha = 1, \dots, n-1$

$$E_i(L^{n,m,k}) = E_{i-\alpha}(L^{n,m+\alpha,k-\alpha}), \qquad i = \alpha + 1, \dots, n, \tag{21}$$

which can also be written as

$$E_i(L^{n,m,k}) = E_{i+\alpha}(L^{n,m-\alpha,k+\alpha}), \qquad i = 1, \dots, n - \alpha. \tag{22}$$

(2)

$$E_l(L^{n,0,2n+\sigma}) = E_{l+1}(L^{n+1,0,2n+\sigma+2}), \qquad \sigma = 1, \dots, n-1, \quad l = \sigma + 1, \dots, n. \tag{23}$$

The proof of this theorem can be found in [1]. This seemingly technical theorem guarantees that the form of Euler–Lagrange equations survives the passage from the n-component to $(n+1)$-component Killing system and hence it will be crucial for the construction of soliton hierarchies below. The index σ will be related to the number of components of the obtained soliton systems.

For our further considerations we will also need a hierarchy of infinite Killing systems

$$\frac{\mathrm{d}}{\mathrm{d}t_i}q_j = (q_{j+i-1})_x + \sum_{k=1}^{j-1}(q_k(q_{j+i-k-1})_x - q_{j+i-k-1}(q_k)_x) \equiv \left(Z_i^\infty[q]\right)^j, \quad i, j = 1, \dots \infty \tag{24}$$

that is known as the universal hierarchy and has been considered in [10].

4. Coupled KdV hierarchies and their solutions

We now briefly remind the reader our specific elimination procedure from [1] that turns the dispersionless Killing systems (19) into cKdV hierarchies. More specifically, we show how to produce s (with $s \in \mathbf{N}$) N-component ($N \in \mathbf{N}$) commuting vector fields (evolutionary systems) by eliminating some variables from a set of Killing systems (19) with the help of Euler–Lagrange equations for an appropriate Lagrangian $L^{n,m,k}$. The crucial for this procedure is that if applied to $s+1$ instead of s it yields the same set of s commuting vector fields plus an extra vector field that commutes with the first s fields. This means that this procedure leads in fact to an infinite hierarchy of commuting vector fields in the sense that for arbitrary s we can produce first s vector fields from the same infinite sequence of commuting vector fields. Moreover it turns out that in this way we produce vector fields with dispersion (soliton systems), namely well-known coupled KdV hierarchies of Antonowicz and Fordy [2] (in a different parametrization). Details are as follows.

Firstly, we choose $A = 1$ and $B = -1$. This specific choice of A and B is introduced only for a smoother identification of our systems with the aforementioned cKdV hierarchies; the elimination procedure works otherwise for arbitrary values of A and B. Consider all N possible splittings

$$N = \sigma + \alpha \qquad \text{with} \quad \sigma \in \{1, \ldots, N\} \quad \text{and} \quad \alpha \in \{0, \ldots, N-1\}.$$

Every such splitting leads to a separate hierarchy. Consider also the Killing systems (19), written in a shorthand way as

$$q_{t_r} = Z_r^n[q_1, \ldots, q_n], \qquad r = 1, \ldots, n, \tag{25}$$

where $q = (q_1, \ldots q_n)^T$.

Remark 7. The first $s = n - N + 1$ equations in (25) are such that their first N components coincide with the corresponding components of the infinite Killing hierarchy (24). The remaining $n - s$ equations in (25) are incomplete with respect to the infinite hierarchy (24) since beginning with the flow $s + 1$ systems (24) contain at its first N components also the variables $q_{n+1}, \ldots, q_{n+N-1}$.

Let us now choose $m = -\alpha$ and $k = 2n + N$ in (14) so that $f = \lambda^{-\alpha}$ and $\gamma = \lambda^{2n+N}$ and consider *the last* $n - N$ Euler–Lagrange equations associated with $L^{n,-\alpha,2n+N}$. One can show [1] that they have the form

$$\begin{aligned}
&E_{N+1}(L^{n,-\alpha,2n+N}) \equiv 2q_n + \varphi_{n-N+1}^{(\alpha)}[q_1, \ldots, q_{n-1}] = 0,\\
&E_{N+2}(L^{n,-\alpha,2n+N}) \equiv 2q_{n-1} + \varphi_{n-N}^{(\alpha)}[q_1, \ldots, q_{n-2}] = 0,\\
&\vdots\\
&E_n(L^{n,-\alpha,2n+N}) \equiv 2q_{N+1} + \varphi_1^{(\alpha)}[q_1, \ldots, q_N] = 0.
\end{aligned} \tag{26}$$

Due to their structure, equations (26) can be explicitly solved with respect to the variables $q_{N+1}, \ldots, q_n$ which yield $q_{N+1}, \ldots, q_n$ as some differential functions of $q_1, \ldots, q_N$:

$$\begin{aligned}
&q_{N+1} = f_1^{(\alpha)}[q_1, \ldots, q_N]\\
&\vdots\\
&q_n = f_{n-N+1}^{(\alpha)}[q_1, \ldots, q_N].
\end{aligned} \tag{27}$$

Naturally, the solutions (8) (with our choice of f and γ) solve both (25) and (26). Thus, *within the class (8) of solutions (13)*, we can use the Euler–Lagrange equations (26) or rather their solved form (27) to successively express (eliminate) the variables $q_{N+1},\ldots,q_n$ as differential functions of $q_1,\ldots,q_N$ in (25). Plugging (27) into (25) produces n vector fields with $N=\sigma+\alpha$ components:

$$\overline{q}_{t_r}=\overline{Z}_r^{n,N,\alpha}[\overline{q}]\qquad r=1,\ldots,n,\quad \alpha\in\{0,\ldots,N-1\}\tag{28}$$

(with $\overline{q}=(q_1,\ldots,q_N)^T$) . The higher components of (25) disappear after this elimination within our class (8) of solutions. Moreover, since the first s equations in (25) are complete in the sense of remark 7 it can be shown that

$$\overline{Z}_r^{n,N,\alpha}[\overline{q}]=\overline{Z}_r^{N,\alpha}[\overline{q}]\qquad r=1,\ldots,s,$$

meaning that the first $s=n-N+1$ equations in (28) do not depend on n. Observe also that equations (26) do not depend on n. Actually, if n increases to n' the last $n-N$ equations in (26) with this new n' will remain unaltered. This means that we can repeat this elimination procedure by taking $n'=n+1$ instead of n (so that s increases to $s+1$ and $k=2n+N$ increases to $2(n+1)+N=k+2$ while σ and α are kept unaltered). This new procedure (with $n'=n+1$ instead of n) will therefore lead to a sequence of $s+1$ autonomous $N=(\sigma+\alpha)$-component systems in which the first s systems will coincide with the corresponding systems obtained from the previous procedure (with s). This way we can obtain arbitrary long sequences of the same infinite set of commuting vector fields (soliton hierarchy):

$$\overline{q}_{t_r}=\overline{Z}_r^{N,\alpha}[\overline{q}]\qquad r=1,2,\ldots\infty,\quad \alpha\in\{0,\ldots,N-1\}\,.\tag{29}$$

The second index α in (29) denotes different hierarchies. It can be shown [1] that the vector fields $\overline{Z}_r^{N,\alpha}$ commute

$$\left[\overline{Z}_i^{N,\alpha},\overline{Z}_j^{N,\alpha}\right]=0\qquad \text{for any}\quad i,j\in\mathbf{N}.$$

Note also that functions $f_i^{(\alpha)}$ in (27) depend on α so that indeed the procedure leads to N different hierarchies. Now, the n functions $\lambda_i(t_1,\ldots,t_n)$ given implicitly by the system of equations

$$t_i+c_i=\pm\frac{1}{2}\sum_{r=1}^{n}\int\frac{\lambda_r^{n-i+\alpha/2}}{\sqrt{\Delta_r^N}}\,\mathrm{d}\lambda_r\qquad i=1,\ldots,n\tag{30}$$

with

$$\Delta_r^N=\lambda_r^{2n+N}+\sum_{j=1}^{n}a_j\lambda_r^{n-j}=\prod_{i=1}^{2n+N}(\lambda_r-E_i)$$

are solutions of the first n equations of the N-component hierarchy (29) with $N=\sigma+\alpha$. The reason is that equations (30) are just equations (8) with $f=\lambda^{-\alpha}$ and $\gamma=\lambda^{2n+N}$ so they clearly solve all equations (28). Moreover, it can be shown that these solutions are zero on $q_{n+1},\ldots,q_{n+N-1}$ expressed as differential functions of $q_1,\ldots,q_N$ through an appropriate system (26) (with $n'=n+N-1$). This means that (30) indeed solve the first n equations in (29).

Consider now the following infinite multi-Lagrangian 'ladder' of Euler–Lagrange equations of the form,

$$E_1(L^{n,m+j-1,k-j+1})=E_2(L^{n,m+j-2,k-j+2})=\cdots=E_n(L^{n,m+j-n,k-j+n})\tag{31}$$

with fixed $m,k\in\mathbf{Z}$ and with $j=\ldots,-1,0,1,\ldots$ (the multi-Lagrangian form of (31) is due to theorem 6). Equations (26) that we use for variable elimination are then a part of this infinite ladder with $m=-\alpha$ and $k=2n+\sigma+\alpha$ and with $j=1,\ldots,n$. Equations (26) are the only equations in the ladder (31) (for this specific choice of m and k) that allow for the elimination described above. All other equations are complicated polynomial differential equations with no obvious structure that do not allow for any elimination procedure. As a consequence, there exist more solutions of the type (30) associated with all the Lagrangians $L^{n,\beta-\alpha,2N-\beta}$ for $\beta = 1, \ldots, n-1$. However, one can show that for $\beta = 2$ the variable q_{n+N-1} is not zero on these solutions so that these relations solve only first $n-1$ equations in (29). More generally, for any $\beta > 1$ the obtained solutions $\lambda_i(t_1, \ldots, t_n)$ will only satisfy first $n - \beta + 1$ equations of the hierarchy (29). We can thus formulate the following theorem.

Theorem 8. *For any $\beta \in \{0, \ldots, n-1\}$ and any $N = \sigma + \alpha < n$ the n functions $\lambda_i(t_1, \ldots, t_n)$ given implicitly by the system of equations*

$$t_i + c_i = \pm\frac{1}{2}\sum_{r=1}^{n}\int \frac{\lambda_r^{n-i+\alpha/2-\beta/2}}{\sqrt{\Delta_r^{(N,\beta)}}}\,\mathrm{d}\lambda_r \qquad i = 1, \ldots, n \tag{32}$$

and with $\Delta_r^{(N,\beta)}$ given by

$$\Delta_r^{(N,\beta)} = \lambda_r^{2n+N-\beta} + \sum_{j=1}^{n} a_j \lambda_r^{n-j} = \prod_{i=1}^{2n+N-\beta} (\lambda_r - E_i)$$

are solutions of the first $n - \beta + 1$ (all n for $\beta = 0, 1$) equations of the N-component hierarchy (29).

The variables $t_1 = x, t_2, \ldots, t_{n-\beta+1}$ in (32) are 'dynamical times' (evolution parameters) of the hierarchy (29) while the variables $t_{n-\beta+2}, \ldots, t_n$ are just free parameters (i.e. the solutions (32) do not solve flows higher than the flow number $n - \beta + 1$). Note also that, due to the structure of (7), all n functions $\lambda_i(t_1, \ldots, t_n)$ obtained in (32) are necessary in order to compute N functions $q_i(t_1, \ldots, t_{n-\beta+1})$ that solve (29). In the case $N = 1$ the solutions (32) are finite-gap solutions for the KdV equation with the parameters E_i playing the role of endpoints of forbidden zones. For $\beta > 1$ these solutions are up to our knowledge new. For $N > 1$ all the solutions (32) are new.

Remark 9. For a fixed $\alpha \in \{0, \ldots, N-1\}$, the following map,

$$\begin{aligned} u_r &= \frac{\partial V_1^{(N,2N)}}{\partial q_{N+1-r}}, && r = 1, \ldots, N-\alpha \\ u_r &= E_{N+1-r}(L^{N,N-\alpha,2N}), && r = N-\alpha+1, \ldots, N \end{aligned} \tag{33}$$

(where $V_1^{(N,2N)}$ denotes the separable potential $V_1^{(2N)}$ in the dimension N) transforms the hierarchy (29) to the hierarchy generated by the spectral problem

$$\left(\lambda^\alpha \partial_x^2 + \sum_{i=1}^{N} u_i \lambda^{N-i}\right)\Psi = \lambda^N \Psi. \tag{34}$$

This is the well-known spectral problem of Antonowicz and Fordy leading to N-component cKdV hierarchies, one for each $\alpha \in \{0, \ldots, N-1\}$. Thus, N hierarchies (29) are nothing else as (reparametrized) N cKdV hierarchies from [2].

5. Zero-energy solutions

Let us now investigate—on a few chosen examples—the nature of solutions (32) in the case that all a_i vanish (zero-energy solutions). In this case the solutions (32) can easily be integrated yielding

$$t_i + c_i = \pm\frac{1}{2-2i-\sigma}\sum_{r=1}^{n}\lambda_r^{1-i-\sigma/2} \qquad i=1,\ldots,n \tag{35}$$

and contain therefore no β (are the same for all $\beta = 0,\ldots,n-1$) and no N except in $n = s+N-1$. Therefore, we obtain the following corollary:

Corollary 10. *For any $N=\sigma+\alpha<n$, the n functions $\lambda_i(t_1,\ldots,t_n)$ given implicitly by the system of equations (35) are solutions of the first n equations of the N-component cKdV hierarchy (29).*

Naturally, the solutions (35) also solve (on the surface $H_i = 0$ for all i) all Stäckel systems (5) for which $f(\lambda)\gamma(\lambda) = \lambda^{2n+\sigma}$. Observe also that the solutions (35) solve all the Euler–Lagrange equations for the infinite sequence of Lagrangians $L^{n,-\alpha+j,2n+N-j}$ where $j\in\mathbf{Z}$. As a consequence, the second part of the map (33) is zero on solutions (35). The reason for this is that due to theorem 6 the expressions

$$E_{N+1-r}(L^{N,N-\alpha,2N}), \qquad r = N-\alpha+1,\ldots,N$$

can be written as

$$E_r(L^{n,0,2n+\sigma}), \qquad r = n-\alpha+1,\ldots,n$$

whereas all the above expressions are members of the sequence $L^{n,-\alpha+j,2n+N-j}$, $j\in\mathbf{Z}$, so that they are zero on solutions (35). This implies that as soon as $\alpha>0$ the solutions (35) in the representation of Antonowicz and Fordy reduce to the solutions of the corresponding hierarchy with the same σ but with $\alpha=0$.

Example 1. We start with the case $N=1$. We wish to obtain the first $s=3$ flows in (29). There is now only one splitting $N=\sigma+\alpha$ possible, namely $\sigma=1$ and $\alpha=0$. This choice leads to the usual KdV hierarchy. We have to take $n=s+N-1=3$. The Killing systems (25) have in this case the form

$$\frac{\mathrm{d}}{\mathrm{d}t_1}\begin{pmatrix}q_1\\q_2\\q_3\end{pmatrix} = \begin{pmatrix}q_{1,x}\\q_{2,x}\\q_{3,x}\end{pmatrix} = Z_1^3$$
$$\frac{\mathrm{d}}{\mathrm{d}t_2}\begin{pmatrix}q_1\\q_2\\q_3\end{pmatrix} = \begin{pmatrix}q_{2,x}\\q_{3,x}+q_1q_{2,x}-q_2q_{1,x}\\q_1q_{3,x}-q_3q_{1,x}\end{pmatrix} = Z_2^3 \tag{36}$$
$$\frac{\mathrm{d}}{\mathrm{d}t_3}\begin{pmatrix}q_1\\q_2\\q_3\end{pmatrix} = \begin{pmatrix}q_{3,x}\\q_1q_{3,x}-q_3q_{1,x}\\q_2q_{3,x}-q_3q_{2,x}\end{pmatrix} = Z_3^3.$$

The Lagrangian $L^{n,-\alpha,2n+\sigma+\alpha} = L^{3,0,7}$ is

$$L^{3,0,7} = \tfrac{1}{4}\left(q_1^2-q_2\right)q_{1,x}^2 - \tfrac{1}{2}q_1q_{1,x}q_{2x} + \tfrac{1}{2}q_{1,x}q_{3,x} + \tfrac{1}{4}q_{2,x}^2 + 2q_2q_3 \\ - 3q_1^2q_3 - 3q_1q_2^2 + 4q_1^3q_2 - q_1^5$$

and the Euler–Lagrange equations (26) attain the form

$$E_2(L^{3,0,7}) \equiv 2q_3 - 6q_1q_2 + 4q_1^3 + \tfrac{1}{4}q_{1,x}^2 + \tfrac{1}{2}q_1q_{1,xx} - \tfrac{1}{2}q_{2,xx} = 0,$$
$$E_3(L^{3,0,7}) \equiv 2q_2 - 3q_1^2 - \tfrac{1}{2}q_{1,xx} = 0.$$

Due to their structure, these equations can be solved with respect to q_2, q_3 yielding (27) of the form

$$q_2 = \tfrac{1}{4}q_{1,xx} + \tfrac{3}{2}q_1^2, \qquad q_3 = \tfrac{1}{16}q_{1,xxxx} + \tfrac{5}{4}q_1q_{1,xx} + \tfrac{5}{8}q_{1,x}^2 + \tfrac{5}{2}q_1^3. \tag{37}$$

Substituting (37) into the Killing systems (36) yields the three one-component flows (29):

$$\begin{aligned}
q_{1,t_1} &= q_{1,x} = \overline{Z}_1^{1,0}\\
q_{1,t_2} &= \tfrac{1}{4}q_{1,xxx} + 3q_1q_{1,x} = \overline{Z}_2^{1,0}\\
q_{1,t_3} &= \tfrac{1}{16}q_{1,xxxxx} + \tfrac{5}{2}q_{1,x}q_{1,xx} + \tfrac{5}{4}q_1q_{1,xxx} + \tfrac{15}{2}q_1^2q_{1,x} = \overline{Z}_3^{1,0},
\end{aligned} \tag{38}$$

which are just the first three flows of the KdV hierarchy. By taking larger s we can produce an arbitrary number of flows from the KdV hierarchy. Now, according to corollary 10, the formula (35) yields some specific solutions of all three flows in (38). Explicitly, this formula reads (with $x = t_1$, $c_i = 0$ and + sign in (35))

$$\begin{aligned}
x &= -\sum_{i=1}^{3} z_i = -\rho_1\\
t_2 &= -\frac{1}{3}\sum_{i=1}^{3} z_i^3 = -\frac{1}{3}\left(\rho_1^3 - 3\rho_1\rho_2 + 3\rho_3\right)\\
t_3 &= -\frac{1}{5}\sum_{i=1}^{3} z_i^5 = -\frac{1}{5}\left(\rho_1^5 - 5(\rho_1\rho_2 - \rho_3)\left(\rho_1^2 - \rho_2\right)\right),
\end{aligned} \tag{39}$$

where $z_i = \lambda_i^{-1/2}$, $i = 1, 2, 3$, and where $\rho_1 = \sum_{i=1}^{3} z_i$, $\rho_2 = z_1z_2 + z_1z_3 + z_2z_3$ and $\rho_3 = z_1z_2z_3$ are elementary symmetric polynomials in z_i. The right-hand sides of (39) follow from Newton formulae:

$$\sum_{i=1}^{n} z_i^m = \sum_{\alpha_1+2\alpha_2+\ldots+n\alpha_n=m} (-1)^{a_2+\alpha_4+\alpha_6+\cdots} m\frac{(\alpha_1+\alpha_2+\cdots+\alpha_n-1)!}{\alpha_1!\ldots\alpha_n!}\rho_1^{\alpha_1}\rho_2^{\alpha_2}\ldots\rho_n^{\alpha_n},$$
$$m = 1, \ldots, n \tag{40}$$

that can easily be extended to the case $m \geqslant n$ by taking larger n and putting all higher ρ_i equal to zero. The system (39) can be solved explicitly yielding the 3-time solutions

$$\begin{aligned}
\rho_1 &= -x\\
\rho_2 &= \frac{15t_3 + 2x^5 - 15x^2t_2}{5(x^3 - 3t_2)}\\
\rho_3 &= \frac{-15t_2x^3 - 45t_2^2 + x^6 + 45xt_3}{15(x^3 - 3t_2)}.
\end{aligned} \tag{41}$$

On the other hand, it is easy to see that

$$q_1 = \left(\frac{\rho_2}{\rho_3}\right)^2 - 2\frac{\rho_1}{\rho_3}. \tag{42}$$

Plugging (41) into (42) we finally obtain a 3-time solution of the first three flows (38) of the KdV hierarchy. It has a rather complicated, rational form:

$$q_1(x,t_2,t_3) = \frac{-3\left(675t_3^2 - 270t_3x^5 + 2x^{10} + 675x^4t_2^2 - 1350xt_2^3\right)}{\left(-15t_2x^3 - 45t_2^2 + x^6 + 45xt_3\right)^2}. \tag{43}$$

If we put $t_3 = 0$ in the solution (43) we obtain a rational solution of the (first) KdV equation as obtained for example in [9] (see formula (3.4) there). Our solution however encompasses also a rational solution for the second KdV flow. By taking larger s we can in this way obtain s-time solutions of first s flows of the KdV hierarchy. The map (33) is in this case trivial and reads $u_1 = 2q_1$.

Example 2. Let us consider the two-field case: $N = 2$. There are now two splittings possible: $N = \sigma + \alpha = 2 + 0$ and $N = \sigma + \alpha = 1 + 1$. We consider only the first two flows $s = 2$ (i.e. only the first nontrivial flow) so that $n = s + N - 1 = 3$ as before and therefore the original Killing systems are as before, i.e (36)—we just consider the first two of them. Let us first consider the splitting $N = \sigma + \alpha = 2 + 0$. The Lagrangian $L^{n,-\alpha,2n+\sigma+\alpha} = L^{3,0,8}$ is

$$L^{3,0,8} = \tfrac{1}{4}\left(q_1^2 - q_2\right) q_{1,x}^2 - \tfrac{1}{2}q_1q_{1,x}q_{2x} + \tfrac{1}{2}q_{1,x}q_{3,x} + \tfrac{1}{4}q_{2,x}^2$$
$$+ q_3^2 - 6q_1q_2q_3 + 4q_3q_1^3 - q_2^3 + 6q_2^2q_1^2 - 5q_2q_1^4 + q_1^6$$

(note that its kinetic energy part is the same as in $L^{3,0,7}$ above). The formulae (26) contain only one equation that can be solved with respect to q_3 yielding (27) of the form

$$q_3 = 3q_1q_2 - 2q_1^3 + \tfrac{1}{4}q_{1,xx}.$$

Substituting this into (25) (with $s = 2$ now) yields the first two flows of the first (i.e with $\alpha = 0$) 2-component cKdV hierarchy:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t_1}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} &= \begin{pmatrix} q_{1,x} \\ q_{2,x} \end{pmatrix} = \overline{Z}_1^{3,0} \\ \frac{\mathrm{d}}{\mathrm{d}t_2}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} &= \begin{pmatrix} q_{2,x} \\ 2q_2q_{1,x} + 4q_1q_{2,x} - 6q_1^2q_{1,x} + \dfrac{1}{4}q_{1,xxx} \end{pmatrix} = \overline{Z}_2^{3,0}. \end{aligned} \tag{44}$$

The zero-energy solutions (35) (again with all $c_i = 0$ and with the plus sign only) attain now the form

$$\begin{aligned} x &= -\frac{1}{2}\sum_{i=1}^{3} z_i = -\rho_1 \\ t_2 &= -\frac{1}{4}\sum_{i=1}^{3} z_i^2 = -\frac{1}{3}\left(\rho_1^3 - 3\rho_1\rho_2 + 3\rho_3\right) \\ t_3 &= -\frac{1}{6}\sum_{i=1}^{3} z_i^3 = -\frac{1}{5}\left(\rho_1^5 - 5(\rho_1\rho_2 - \rho_3)(\rho_1^2 - \rho_2)\right), \end{aligned} \tag{45}$$

where ρ_i again denote elementary symmetric polynomials in z_i but where now $z_i = \lambda_i^{-1}$. Solving (45) yields the following 3-time solutions:

$$\rho_1 = -2x \qquad \rho_2 = 2x^2 + 2t_2 \qquad \rho_3 = -\frac{4}{3}x^3 - 4xt_2 - 2t_3, \tag{46}$$

where the variable t_3 plays the role of a free parameter for equations (44). Moreover, we have

$$q_1 = -\frac{\rho_2}{\rho_3}, \qquad q_2 = \frac{\rho_1}{\rho_3}. \tag{47}$$

Plugging (46) into (47) we obtain the following solutions for (44):

$$q_1(x,t_2,t_3)=\frac{3(t_2+x^2)}{3t_3+2x^3+6xt_2},\qquad q_2(x,t_2,t_3)=\frac{3x}{3t_3+2x^3+6xt_2}.\qquad (48)$$

Note that corollary 10 implies that the functions (48) solve first $n=3$ flows of the hierarchy (29) with $N=2$ and $\alpha=0$. In order to compute this third flow we just need to take $s=3$ in the elimination procedure. The result is

$$\frac{\mathrm{d}}{\mathrm{d}t_3}\begin{pmatrix}q_1\\ q_2\end{pmatrix}=\begin{pmatrix}3q_2q_{1,x}+3q_1q_{2,x}-6q_1^2q_{1,x}+\frac{1}{4}q_{1,xxx}\\ 6q_1q_2q_{1,x}-18q_1^3q_{1,x}+6q_1^2q_{2,x}+\frac{3}{4}q_1q_{1,xxx}+3q_2q_{2,x}+\frac{1}{4}q_{2,xxx}\end{pmatrix}=\overline{Z}_3^{3,0}.$$

Let us finally pass to Antonowicz–Fordy variables. The map (33) attains the form

$$u_1=2q_1,\qquad u_2=2q_2-3q_1^2$$

and it transforms both systems in (44) to the representation of Antonowicz and Fordy. Explicitly

$$\begin{gathered}\frac{\mathrm{d}}{\mathrm{d}t_1}\begin{pmatrix}u_1\\ u_2\end{pmatrix}=\begin{pmatrix}u_{1,x}\\ u_{2,x}\end{pmatrix}=\overline{Z}_1^{3,0}[u]\\ \frac{\mathrm{d}}{\mathrm{d}t_2}\begin{pmatrix}u_1\\ u_2\end{pmatrix}=\begin{pmatrix}u_{2,x}+\frac{3}{2}u_1u_{1,x}\\ u_2u_{1,x}+\frac{1}{2}u_1u_{2,x}+\frac{1}{4}u_{1,xxx}\end{pmatrix}=\overline{Z}_2^{3,0}[u].\end{gathered}\qquad (49)$$

In the u-variables the solutions (48) yield solutions for (49) and attain the form

$$u_1(x,t_2,t_3)=\frac{6(t_2+x^2)}{3t_3+2x^3+6xt_2},\qquad u_2(x,t_2,t_3)=\frac{3\left(6xt_3-5x^4-6x^2t_2-9t_2^2\right)}{(3t_3+2x^3+6xt_2)^2}.$$

Example 3. Let us now consider the case $N=\sigma+\alpha=1+1$. Again, we look for the first $s=2$ flows in (29) with $N=2$, $a=1$. We have to take $n=s+N-1=3$ and the rather lengthy Lagrangian

$$\begin{aligned}L^{n,-\alpha,2n+\sigma+\alpha}=L^{3,-1,8}&=-\tfrac{1}{4}\left(q_1^3-2q_1q_2+q_3\right)q_{1,x}^2+\tfrac{1}{2}\left(q_1^2-q_2\right)q_{1,x}q_{2,x}\\&\quad-\tfrac{1}{2}q_1q_{1,x}q_{3,x}-\tfrac{1}{4}q_1q_{2,x}^2+\tfrac{1}{2}q_{2,x}q_{3,x}\\&\quad+q_3^2-6q_1q_2q_3+4q_3q_1^3-q_2^3+6q_2^2q_1^2-5q_2q_1^4+q_1^6\end{aligned}$$

(the potential part is of course the same as in $L^{3,0,8}$ above). The elimination equations (26) yield again only one equation that being solved with respect to q_3 reads

$$q_3=-\tfrac{1}{8}q_{1,x}^2-2q_1^3+3q_1q_2-\tfrac{1}{4}q_1q_{1,xx}+\tfrac{1}{4}q_{2,xx}.$$

Plugging this into the two first flows in (36) we obtain the first two flows of the second (i.e. with $\alpha=1$) 2-field cKdV hierarchy:

$$\begin{gathered}\frac{\mathrm{d}}{\mathrm{d}t_1}\begin{pmatrix}q_1\\ q_2\end{pmatrix}=\begin{pmatrix}q_{1,x}\\ q_{2,x}\end{pmatrix}=\overline{Z}_1^{3,1}\\ \frac{\mathrm{d}}{\mathrm{d}t_2}\begin{pmatrix}q_1\\ q_2\end{pmatrix}=\begin{pmatrix}q_{2,x}\\ 2q_2q_{1,x}+4q_1q_{2,x}-\frac{1}{2}q_{1,x}q_{1,xx}-6q_1^2q_{1,x}-\frac{1}{4}q_1q_{1,xxx}+\frac{1}{4}q_{2,xxx}\end{pmatrix}=\overline{Z}_2^{3,1}\end{gathered}$$

(50)

Now, the solutions (35) in this case attain precisely the form (39), or (41) in solved form, since both n and σ are the same in both cases. However, (50) are two-component, so in this case we need to express both q_1 and q_2 as functions of ρ_i,

$$q_1=\left(\frac{\rho_2}{\rho_3}\right)^2-2\frac{\rho_1}{\rho_3},\qquad q_2=\left(\frac{\rho_1}{\rho_3}\right)^2-2\frac{\rho_2}{\rho_3^2}.\qquad (51)$$

Substituting (41) into (51) we finally arrive at a 3-time solution of (50) with t_3 as a free parameter:

$$q_1(x,t_2,t_3) = \frac{-3\left(675t_3^2 - 270t_3x^5 + 2x^{10} + 675x^4t_2^2 - 1350xt_2^3\right)}{\left(-15t_2x^3 - 45t_2^2 + x^6 + 45xt_3\right)^2},$$
$$q_2(x,t_2,t_3) = \frac{45(x^3-3t_2)(x^5+15x^2t_2-30t_3)}{\left(-15t_2x^3 - 45t_2^2 + x^6 + 45xt_3\right)^2}. \tag{52}$$

As in the previous example, the above solutions solve first $n = 3$ flows of this cKdV hierarchy which means that they also solve the next flow in the hierarchy (with the dynamical time t_3). We will however not write it here. Finally, let us pass to the Antonowicz–Fordy representation. The map (33) is now

$$u_1 = 2q_1, \qquad u_2 = 2q_2 - 3q_1^2 - \tfrac{1}{2}q_{1,xx}$$

so that (50) in Antonowicz–Fordy variables reads

$$\frac{\mathrm{d}}{\mathrm{d}t_1}\begin{pmatrix}u_1\\u_2\end{pmatrix} = \begin{pmatrix}u_{1,x}\\u_{2,x}\end{pmatrix} = \overline{Z}_1^{3,1}[u]$$
$$\frac{\mathrm{d}}{\mathrm{d}t_2}\begin{pmatrix}u_1\\u_2\end{pmatrix} = \begin{pmatrix}u_{2,x} + \frac{3}{2}u_1u_{1,x} + \frac{1}{4}u_{1,xxx}\\ u_2u_{1,x} + \frac{1}{2}u_1u_{2,x}\end{pmatrix} = \overline{Z}_2^{3,1}[u]. \tag{53}$$

In the u-variables the solutions (52) yield solutions for (53) and attain the form

$$u_1(x,t_2,t_3) = \frac{-6\left(675t_3^2 - 270t_3x^5 + 2x^{10} + 675x^4t_2^2 - 1350xt_2^3\right)}{\left(-15t_2x^3 - 45t_2^2 + x^6 + 45xt_3\right)^2}, \qquad u_2(x,t_2,t_3) = 0$$

which is nothing but the solution (43) of the first three flows of the KdV hierarchy (with $N = 1$), in accordance with the observation at the beginning of this section.

Remark 11. It can be shown that our method yields rational solutions only for $\sigma = 1$ and $\sigma = 2$. For $\sigma > 2$ our method leads to new *implicit* solutions of our cKdV hierarchies .

Example 4. Let us thus finally investigate the case $N = 3 = \sigma + \alpha = 3 + 0$ that will lead to implicit solutions. We take again $s = 2$ so that $n = s + N - 1 = 4$. The first $s = 2$ Killing systems in (25) now have the form

$$\frac{\mathrm{d}}{\mathrm{d}t_1}\begin{pmatrix}q_1\\q_2\\q_3\\q_4\end{pmatrix} = \begin{pmatrix}q_{1,x}\\q_{2,x}\\q_{3,x}\\q_{4,x}\end{pmatrix} = Z_1^4$$
$$\frac{\mathrm{d}}{\mathrm{d}t_2}\begin{pmatrix}q_1\\q_2\\q_3\\q_4\end{pmatrix} = \begin{pmatrix}q_{2,x}\\ q_{3,x} + q_1q_{2,x} - q_2q_{1,x}\\ q_{4,x} + q_1q_{3,x} - q_3q_{1,x}\\ q_1q_{4,x} - q_4q_{1,x}\end{pmatrix} = Z_2^4. \tag{54}$$

The Lagrangian $L^{n,-\alpha,2n+N} = L^{4,0,11}$ yields one elimination equation (26):

$$E_4(L^{4,0,11}) = 2q_4 + 12q_2q_1^2 - 6q_1q_3 - 3q_2^2 - 5q_1^4 - \tfrac{1}{2}q_{1,xx} = 0.$$

Solving this with respect to q_4 we obtain

$$q_4 = -6q_2q_1^2 + 3q_1q_3 + \tfrac{3}{2}q_2^2 - \tfrac{5}{2}q_1^4 + \tfrac{1}{4}q_{1,xx}.$$

Substituting this into (54) yields two first flows of the 3-component cKdV hierarchy (29) with $\alpha = 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t_1}\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} q_{1,x} \\ q_{2,x} \\ q_{3,x} \end{pmatrix} = \overline{Z}_1^{3,0}$$
$$\frac{\mathrm{d}}{\mathrm{d}t_2}\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} q_{2,x} \\ q_{3,x} + q_1 q_{2,x} - q_2 q_{1,x} \\ 2q_3q_{1,x} + 4q_1q_{3,x} - 6q_1^2q_{2,x} - 12q_1q_2q_{1,x} + 3q_2q_{2,x} + 10q_1^3q_{1,x} + \frac{1}{4}q_{1,xxx} \end{pmatrix} = \overline{Z}_2^{3,0}. \tag{55}$$

The solutions (35) are now (again with all $c_i = 0$ and with the plus sign only):

$$\begin{aligned} x &= -\frac{1}{3}\sum_{i=1}^{4} z_i^3 = -\frac{1}{3}\left(\rho_1^3 - 3\rho_1\rho_2 + 3\rho_3\right) \\ t_2 &= -\frac{1}{5}\sum_{i=1}^{4} z_i^5 = -\frac{1}{5}\left(\rho_1^5 - 5(\rho_1\rho_2 - \rho_3)\left(\rho_1^2 - \rho_2\right)\right) \\ t_3 &= -\frac{1}{7}\sum_{i=1}^{4} z_i^7 = -\frac{1}{7}\left(\rho_1^7 - 7(\rho_1\rho_2 - \rho_3)\left(\left(\rho_1^2 - \rho_2\right)^2 + \rho_1\rho_3\right) - 7\rho_4\left(\rho_1^3 - 2\rho_1\rho_2 + \rho_3\right)\right) \\ t_4 &= -\frac{1}{9}\sum_{i=1}^{4} z_i^9 = -\frac{1}{9}P_9 \end{aligned} \tag{56}$$

with $z_i = \lambda_i^{-1/2}$ where P_9 is a complicated polynomial of degree nine in ρ_i that can be obtained from the Newton formulae (40). This system cannot be algebraically solved with respect to ρ_i. However, if we embed the system (56) in the system

$$\begin{aligned} \alpha &= -\sum_{i=1}^{5} z_i = -\rho_1 \\ x &= -\frac{1}{3}\sum_{i=1}^{5} z_i^3 = -\frac{1}{3}\left(\rho_1^3 - 3\rho_1\rho_2 + 3\rho_3\right) \\ t_2 &= -\frac{1}{5}\sum_{i=1}^{5} z_i^5 = -\frac{1}{5}\left(\rho_1^5 - 5(\rho_1\rho_2 - \rho_3)\left(\rho_1^2 - \rho_2\right)\right) + 5\rho_1\rho_4 + 5\rho_5 \\ t_3 &= -\frac{1}{7}\sum_{i=1}^{5} z_i^7 = -\frac{1}{7}\left(\rho_1^7 - 7(\rho_1\rho_2 - \rho_3)\left(\left(\rho_1^2 - \rho_2\right)^2 + \rho_1\rho_3\right)\right. \\ &\qquad \left. - 7\rho_4\left(\rho_1^3 - 2\rho_1\rho_2 + \rho_3\right)\right) + 7\rho_1^2\rho_5 \\ t_4 &= -\frac{1}{9}\sum_{i=1}^{5} z_i^9 = -\frac{1}{9}Q_9 \end{aligned} \tag{57}$$

where α is a parameter, and where Q_9 is a polynomial of degree nine in $\rho_1, \ldots, \rho_5$ such that $Q_9|_{\rho_5=0} = P_9$, then obviously the solution of (57) with the condition $\rho_5 = 0$ will yield the solution for (56). The system (57) is algebraically solvable and yields

$$\rho_i = R_i(\alpha, x, t_2, t_3, t_4), \tag{58}$$

where R_i are complicated rational functions of their arguments. The polynomial equation $\rho_5 = 0$ yields then implicitly a (multivalued) function $\alpha = f(x, t_2, t_3, t_4)$. Now, using the fact that

$$q_1 = -\left(\frac{\rho_3}{\rho_4}\right)^2 + 2\frac{\rho_2}{\rho_4}, \qquad q_2 = \left(\frac{\rho_2}{\rho_4}\right) - 2\frac{\rho_1\rho_3}{\rho_4^2} + \frac{2}{\rho_4},$$
$$q_3 = -\frac{\left(\rho_1^4 + 2\rho_2^2 - 4\rho_1^2\rho_2 + 4\rho_1\rho_3 - 4\rho_4\right)}{\rho_4^2}$$

we arrive at an implicit solution of (55) of the form

$$\rho_5(\alpha, x, t_2, t_3, t_4) = 0, \qquad q_i = r_i\,(\alpha, x, t_2, t_3, t_4)\,, \qquad i = 1, 2, 3$$

that is thus determined up to the implicitly expressed function $\alpha = f(x, t_2, t_3, t_4)$. The concrete formulae have been obtained with the help of Maple and are too complicated to present them here. The map (33) does not simplify these solutions.

6. Conclusions

In this paper we presented a method of constructing coupled Korteweg–de Vries hierarchies from the Benenti class of Stäckel separable systems. Our method allows for producing certain classes of solutions of these hierarchies from solutions of corresponding Stäckel systems (theorem 8 and corollary 10). For $N = 1$ and for $\beta = 0$ and $\beta = 1$ we obtain in general finite-gap solutions of KdV. In the zero-energy case and for $N = 1$ we arrive at the known formulae for rational solutions of KdV. It is well known that rational solutions of KdV originate as asymptotics (in a long-wave limit, as the wavenumber $k \to 0$) of multi-soliton solutions [7]. We show here that these solutions are also asymptotic solutions of finite-gap solutions in the zero-energy limit, i.e. when all a_i are zero. It is not clear for us at the moment if our method can also explicitly produce soliton solutions. Similar remarks apply to the case $N > 1$, where our solutions are new. For $\sigma > 2$ all our solutions are implicit in the sense described in example 4.

It is interesting to ask whether the obtained solutions are somehow related to some class of symmetry reductions. Some hint is given in the case $N = 1$: the finite gap solutions (32) for $\beta = 0$ and $\beta = 1$ for the KdV can be obtained from its stationary flows constructed with the help of first two local Hamiltonian representations of the KdV hierarchy. This question is beyond the scope of this paper but it certainly deserves a separate study.

It has to be stressed that our method is general in the sense that other separation relations lead to other hierarchies like, for example, coupled Harry–Dym hierarchies as well as to some specific solutions of these hierarchies. These issues will be studied in a separate paper.

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